Error in the Euclidean Preference Model

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👤= individuals ⚪ = alternatives

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Preferences

	👤	👤	👤	👤
⚪	1	4	2	3
⚪	3	2	3	1
⚪	2	3	3	2
⚪	4	1	4	5
⚪	5	5	1	4

👤 = individuals ⚪ = alternatives

Preferences

	👤	👤	👤	👤	⋯
⚪	1	4	2	3	⋯
⚪	3	2	3	1	⋯
⚪	2	3	5	2	⋯
⚪	4	1	4	5	⋯
⚪	5	5	1	4	⋯
⋮	⋮	⋮	⋮	⋮	⋱

I individuals

A alternatives

👤= individuals ⚪ = alternatives

Preferences

I individuals

	👤	👤	👤	👤	⋯
⚪	15002	283	28319	4354	⋯
⚪	9843	10	2928	47382	⋯
⚪	48616	92830	2935	18237	⋯
⚪	20394	293	72892	27493	⋯
⚪	86	5632	8883	3	⋯
⋮	⋮	⋮	⋮	⋮	⋱

A alternatives

👤 = individuals ⚪ = alternatives

Preferences

	👤	👤	👤	👤	⋯
⚪	15002	283	28319	4354	⋯
⚪	9843	10	2928	47382	⋯
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⋮	⋮	⋮	⋮	⋮	⋱

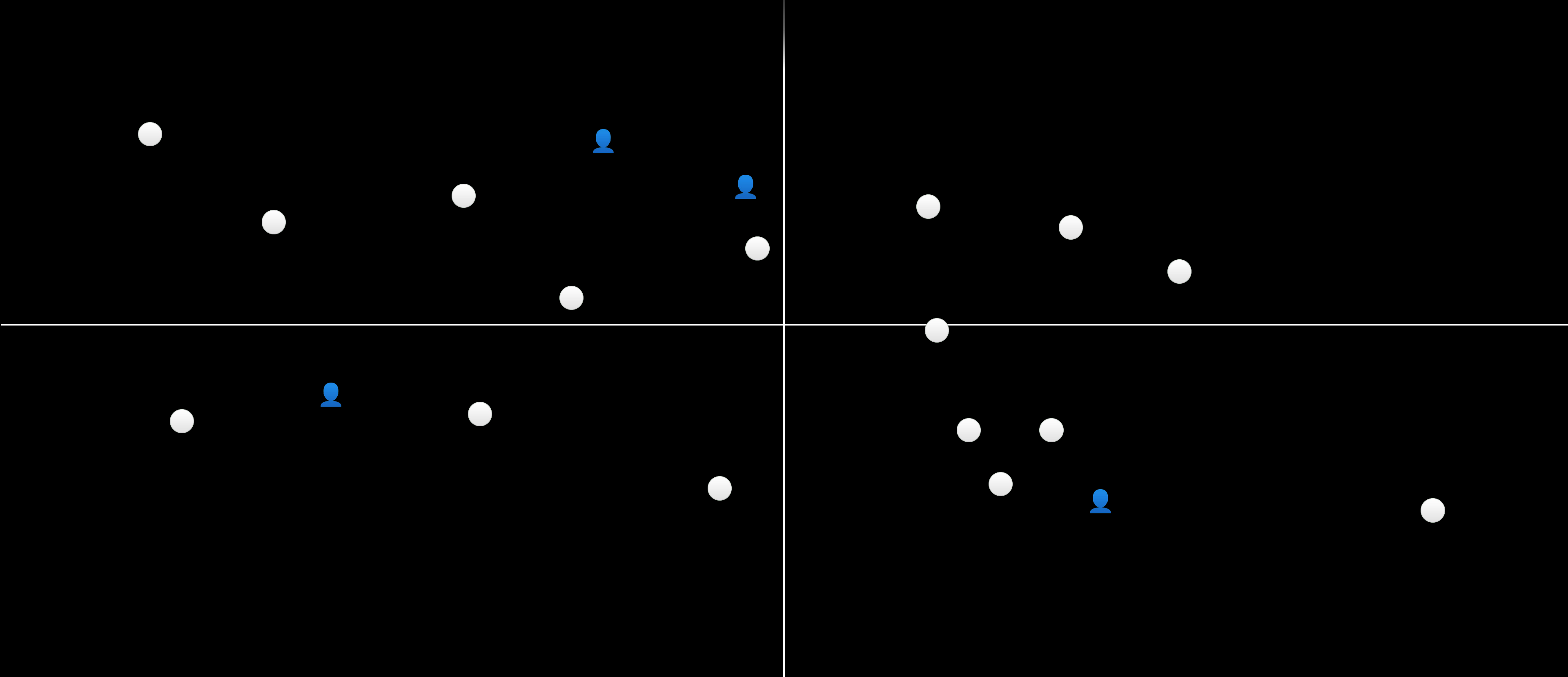
I individuals

A alternatives

preference profile

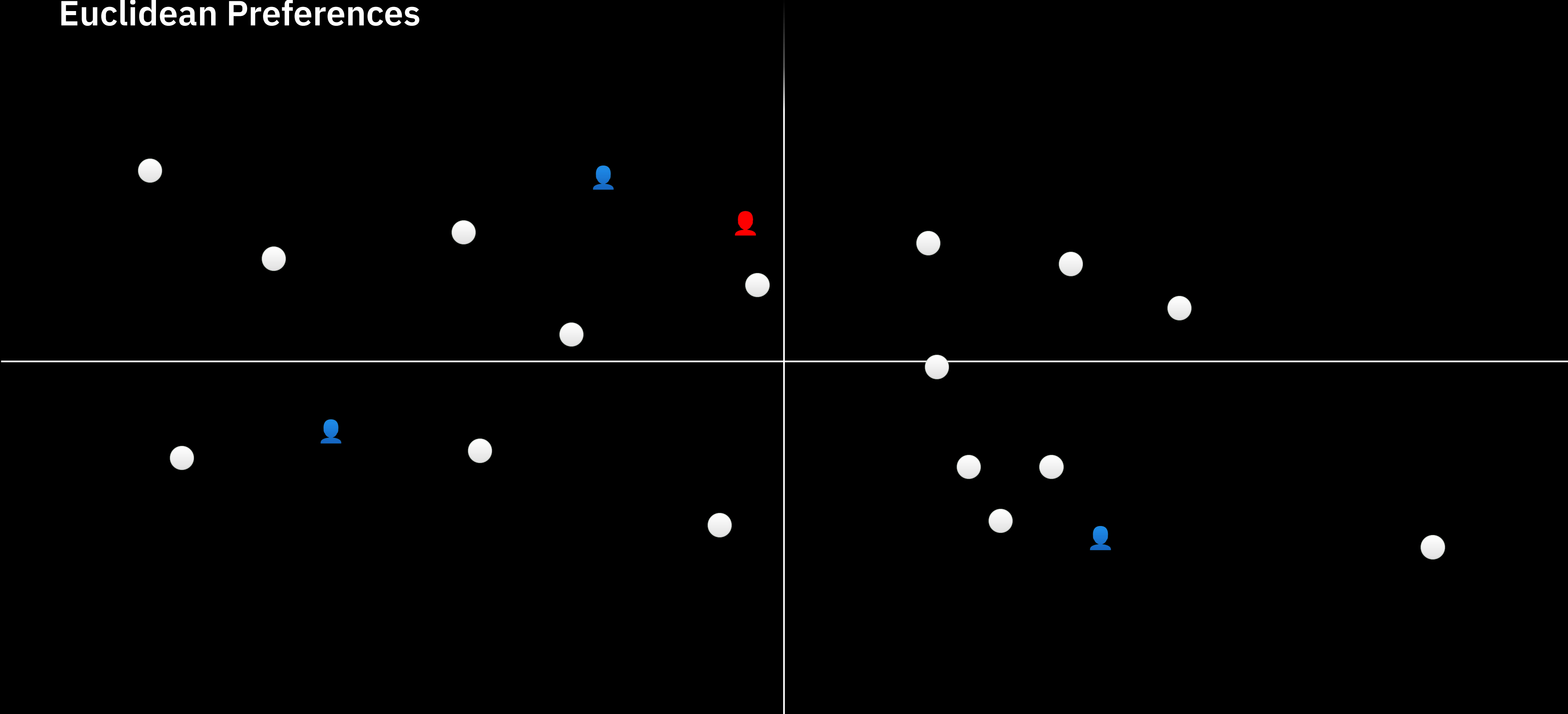
👤 = individuals ⚪ = alternatives

Euclidean Preferences

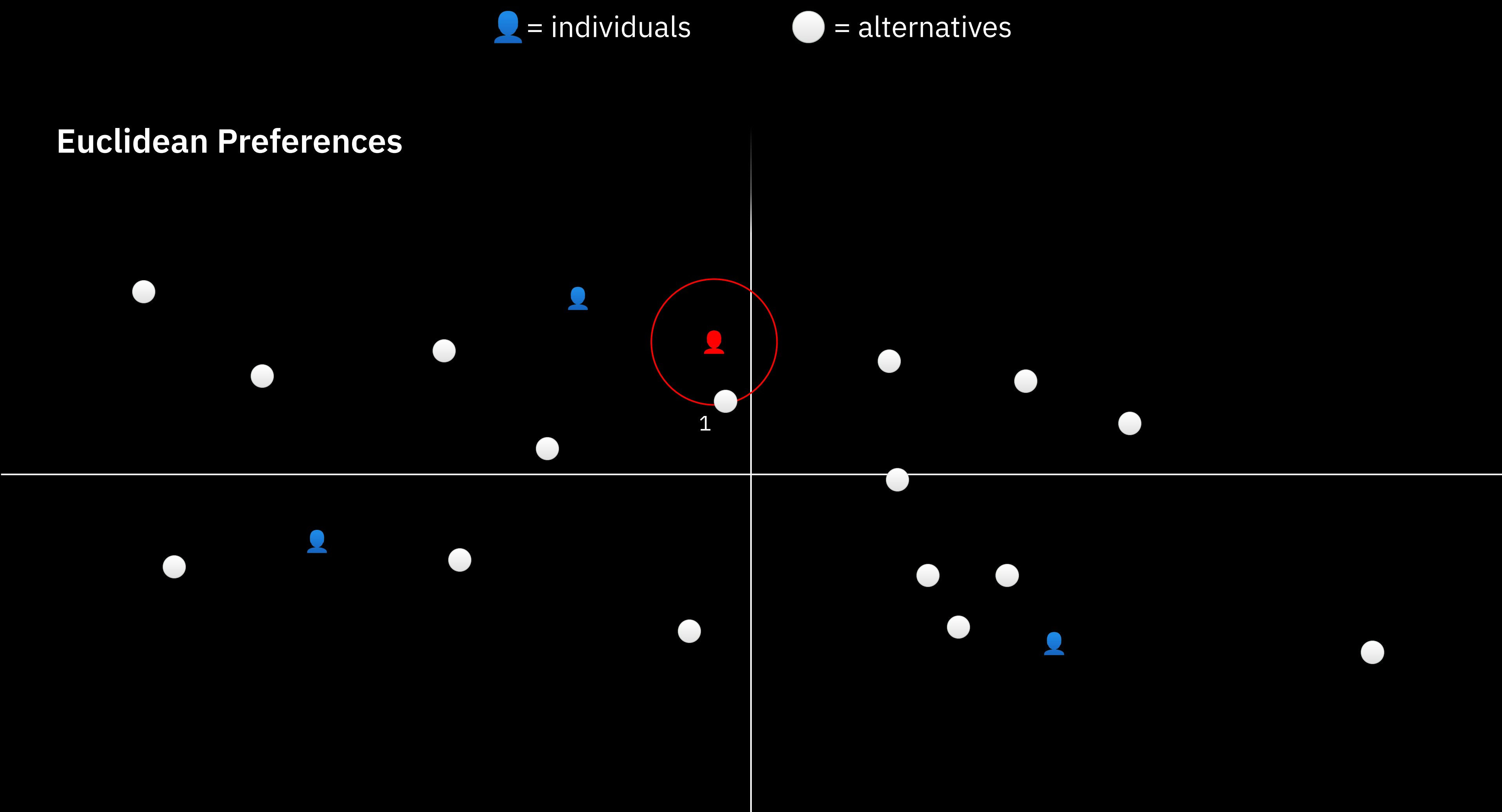


👤 = individuals ⚪ = alternatives

Euclidean Preferences

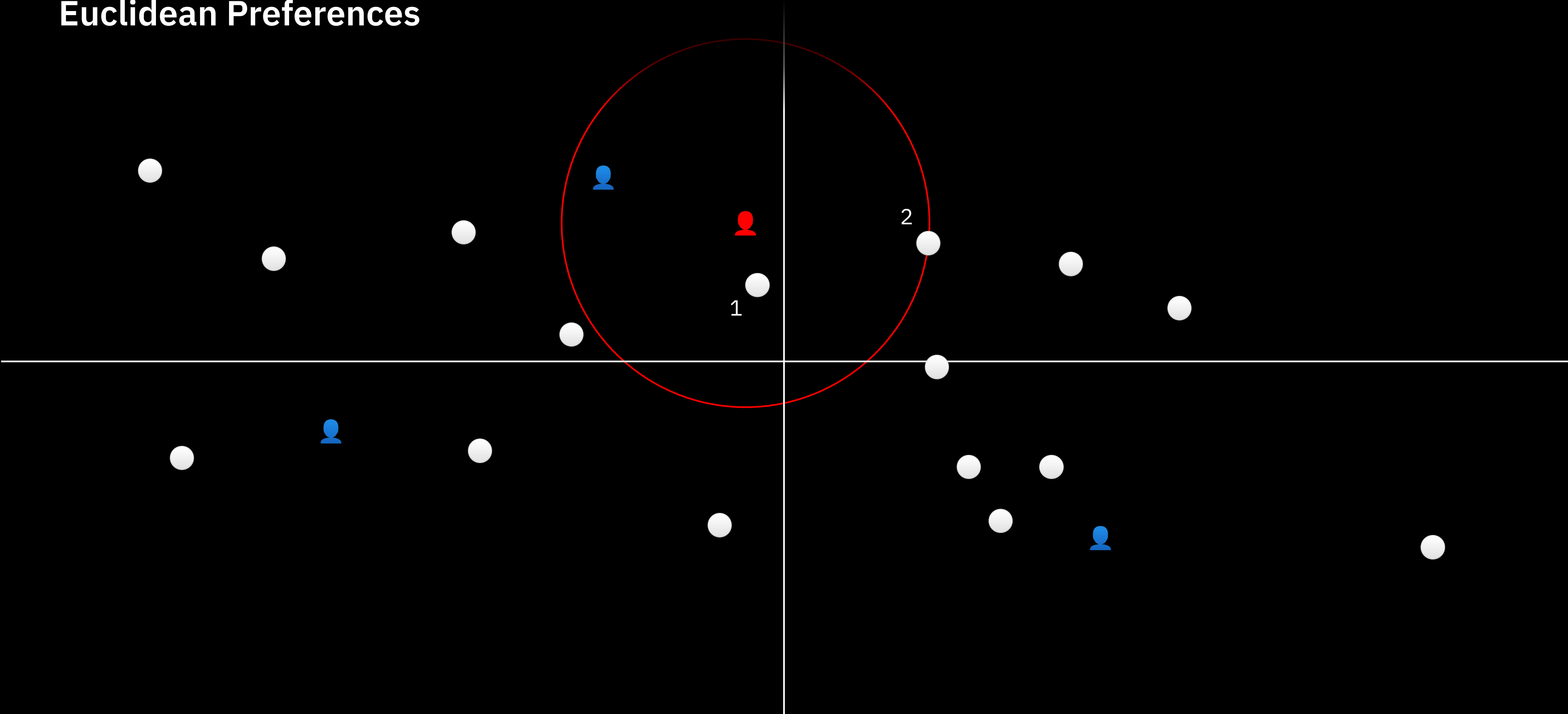


Euclidean Preferences

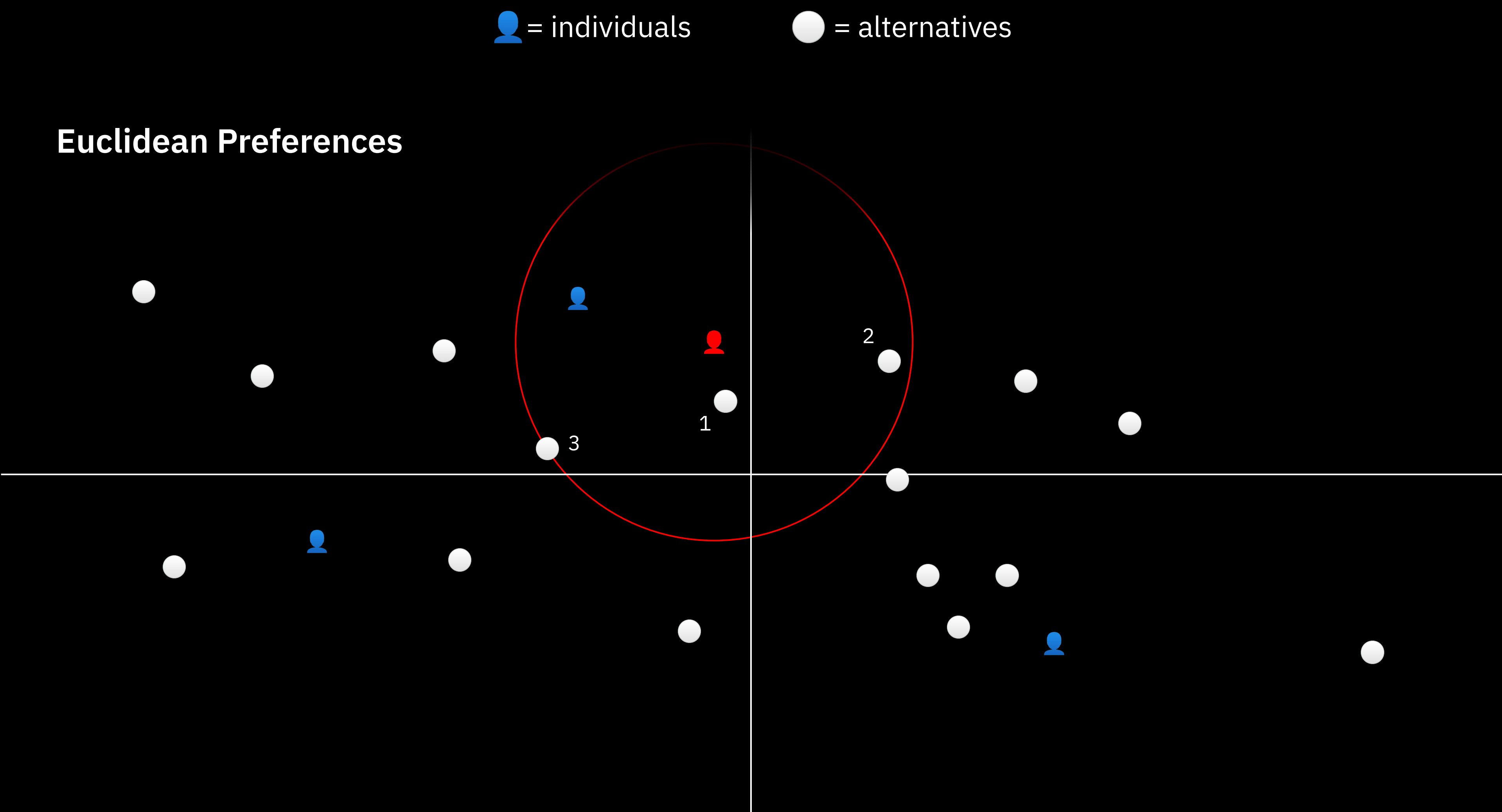


👤 = individuals ⚪ = alternatives

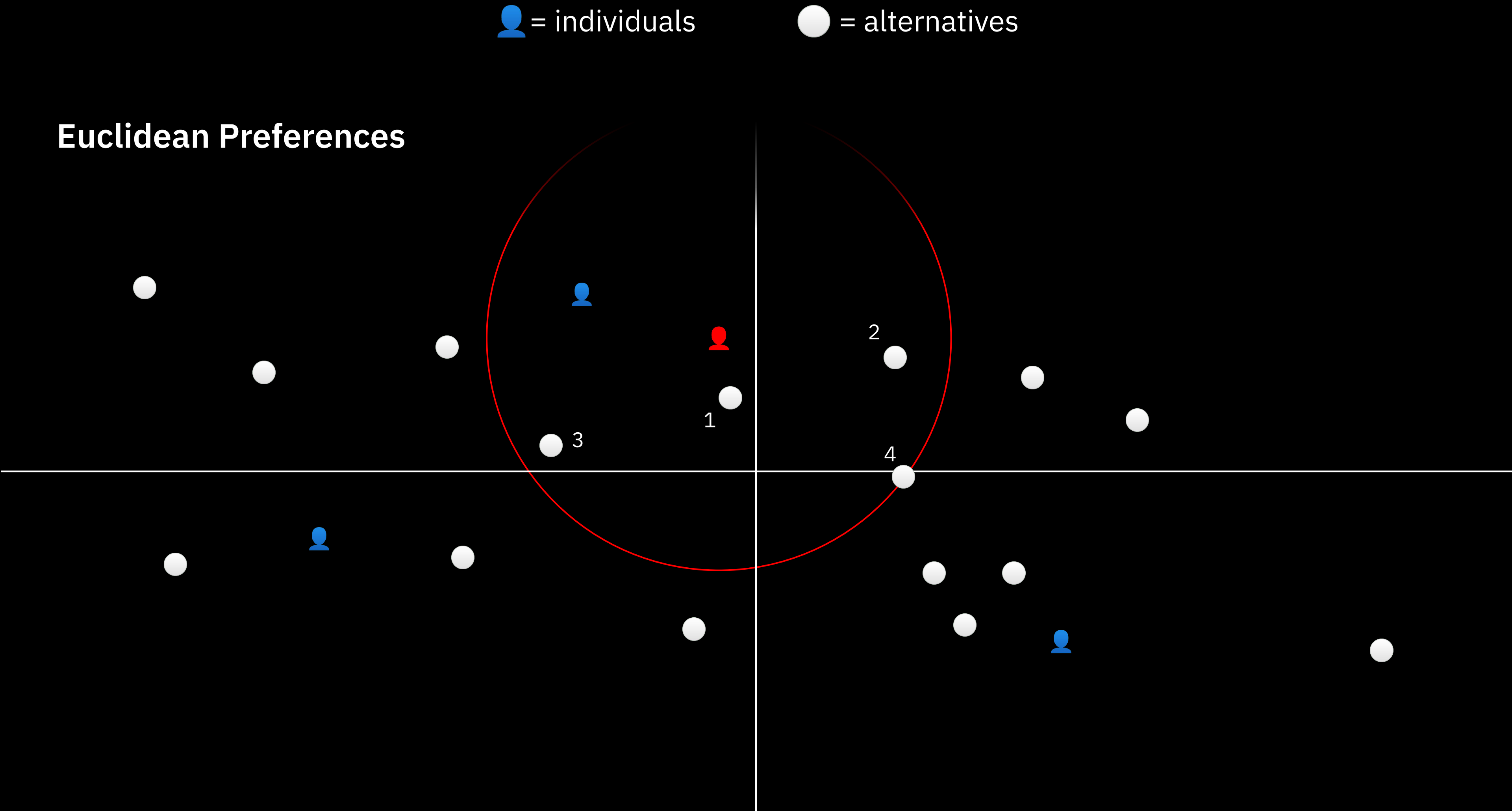
Euclidean Preferences



Euclidean Preferences

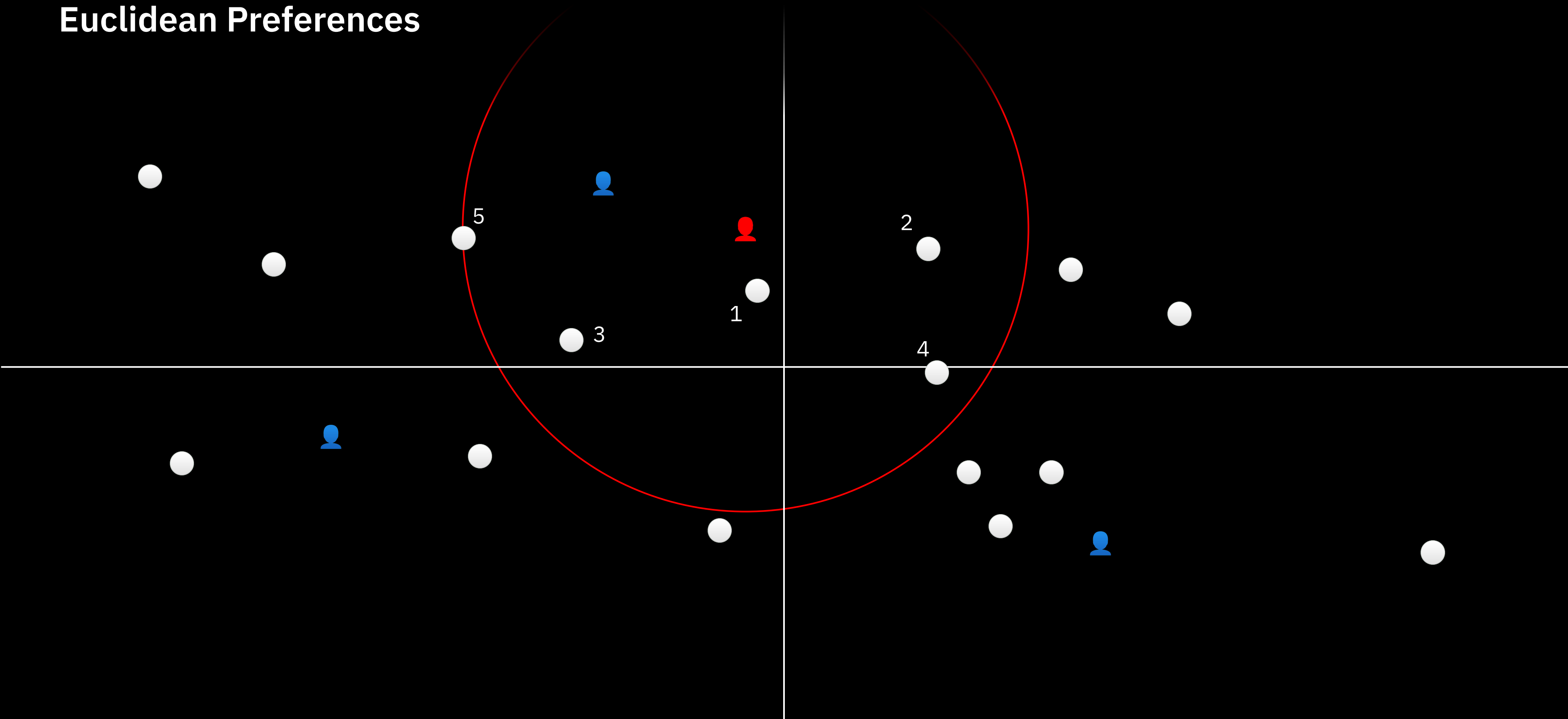


Euclidean Preferences



👤 = individuals ⚪ = alternatives

Euclidean Preferences



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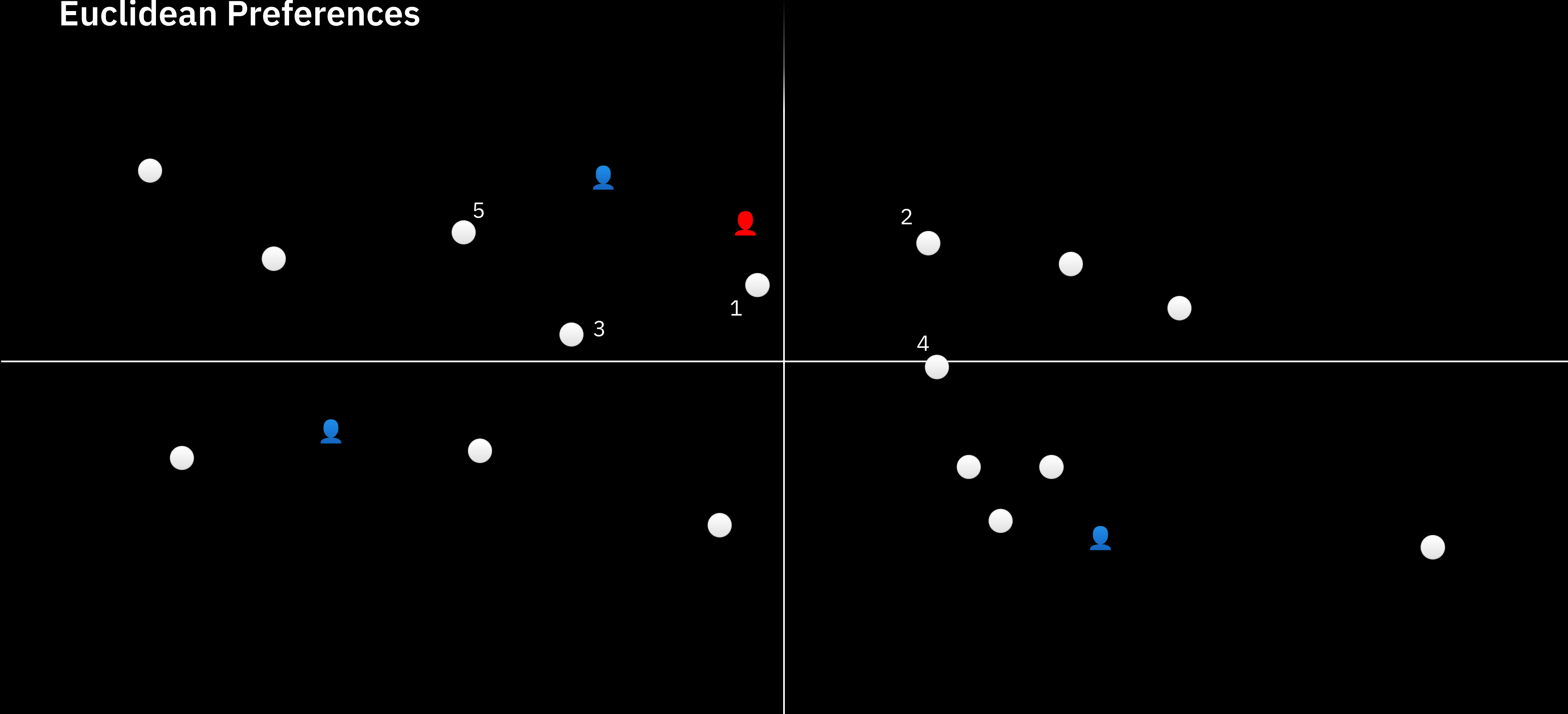
Euclidean Preferences



1. Are all preference profiles Euclidean?

2. What proportion of preference profiles are non-Euclidean?

3. How much information do we lose when shoehorning non-Euclidean profiles into Euclidean space?

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No, not if $\boldsymbol{d} \lesssim \min(\boldsymbol{A}, \boldsymbol{I})$

→ Euclidean Preferences
Bogomolnaia + Laslier (2007)

$$
\begin{array}{ccccccccc}
a_1 & >_1 & a_2 & >_1 & \cdots & >_1 & a_{k-1} & >_1 & a_k \\
a_2 & >_2 & a_3 & >_2 & \cdots & >_2 & a_k & >_2 & a_1 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_k & >_k & a_1 & >_k & \cdots & >_k & a_{k-2} & >_k & a_{k-1}
\end{array}
$$

circulant pathology

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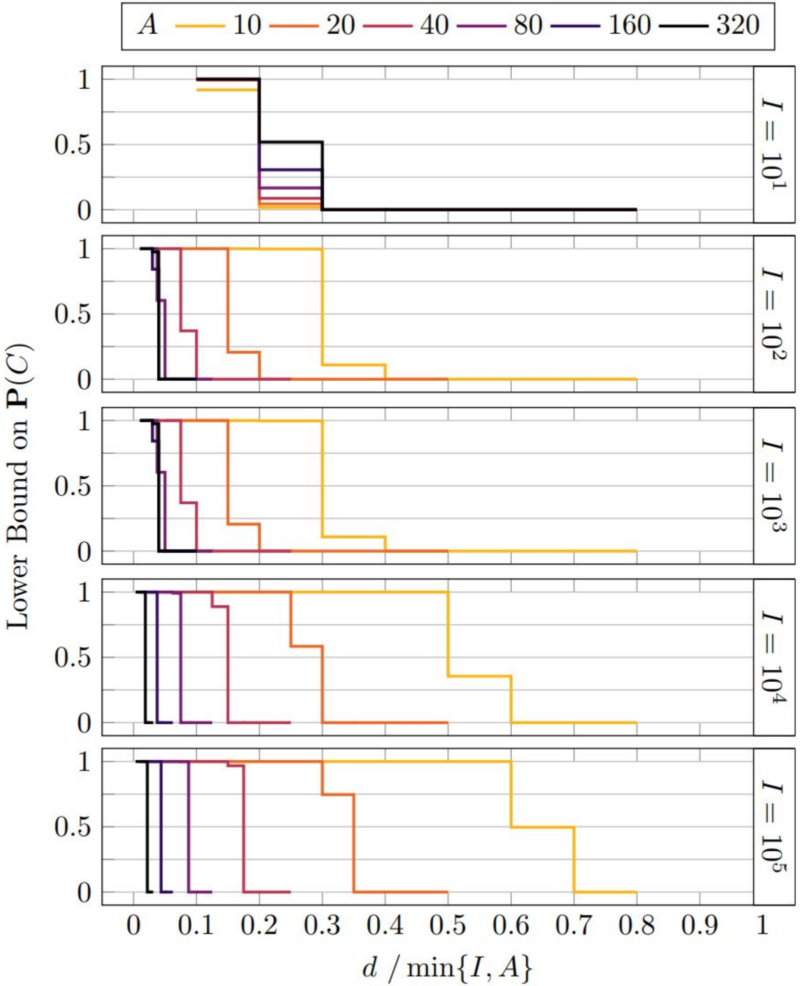
Theorem 3 (lower bound on probability of circulant pathology). *Let A, I, and d be fixed positive integers such that $d < \min\{I, A-1\}$, and $\mathbf{P}(C)$ be the probability that a profile chosen uniformly from $\mathcal{P}_{A,I}$ contains a circulant pathology of size $k \geq d+2$. Then,*

$$\mathbf{P}(C) \geq 1 - \left(1 - \sum_{k=d+2}^{I} B_k\right)^{\left\lfloor \frac{A}{d+2} \right\rfloor},$$

where $B_k = \binom{I}{k} \left\{ {k \atop d+2} \right\} (d+2)! \left(\frac{1}{(d+2)!}\right)^k \left(1 - \frac{d+2}{(d+2)!}\right)^{I-k}$ *and* $\left\{ {k \atop d+2} \right\}$ *denotes a Stirling number of the second kind.*



A
10
20
40
80
160
320
Lower Bound on $\mathbf{P}(C)$
d / min$\{I, A\}$
$I = 10^1$
$I = 10^2$
$I = 10^3$
$I = 10^4$
$I = 10^5$
0
0.5
1
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

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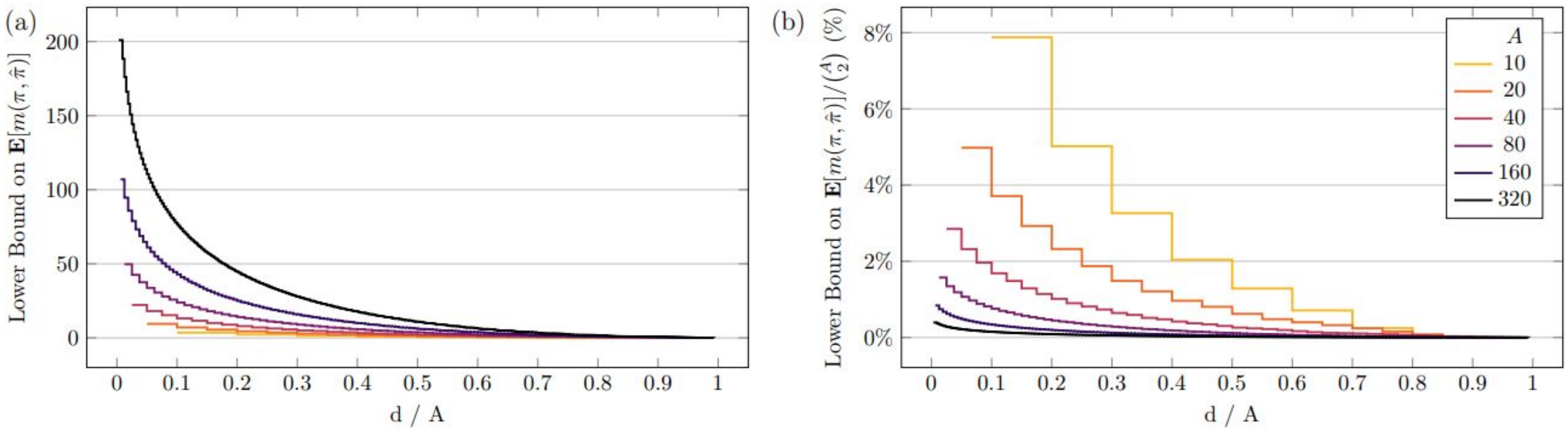
Theorem 4 (lower bound on expected error)**.** *Let* A*,* d *be fixed positive integers such that* $d < A-1$*,* $\Pi \in \mathcal{P}_{A,I^*}$ *consist of* I^* *unique preferences,* π *be a preference chosen uniformly at random from the set of* $A!$ *possible preferences,* $\hat{\pi}$ *be the nearest preference to* π *that is representable in Euclidean*(Π) *(that is, the representable preference that can be reached in the fewest number of adjacent swaps), and* K *be a positive integer such that* $K \leq \binom{A}{2}$*. If* $I^* \geq r$*, then*

$$\mathbf{E}[m(\pi, \hat{\pi})] \geq \sum_{k=0}^{K} \frac{(A! - n_{k,A})_{\hat{r}}}{(A!)_{\hat{r}}} \mathbf{1}(\hat{r} < A! - n_{k,A}),$$

where $(\cdot)_{\hat{r}}$ *denotes a falling factorial,* $\mathbf{1}(\cdot)$ *the indicator function, and* $n_{k,A} = \min\{(A-1)^k,\ A!\}$*.*



(a)
Lower Bound on $\mathbf{E}[m(\pi, \hat{\pi})]$
200
150
100
50
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
d / A
(b)
Lower Bound on $\mathbf{E}[m(\pi, \hat{\pi})]/\binom{A}{2}$ (%)
8%
6%
4%
2%
0%
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
d / A
A
10
20
40
80
160
320

IMPLICATIONS FOR EMBEDDINGS

cosine distance

SoftMax

MAIN LIMITATIONS

1. Bounds are not tight.
2. Only applicable when $I^* \geq r$.
3. Expected error assumes impartial culture.

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